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A note on background independence in noncommutative gauge theories, matrix model, and tachyon condensation

Nathan Seiberg

School of Natural Sciences, Institute for Advanced Study
Einstein Drive, Princeton, NJ 08540, USA
E-mail: seiberg@ias.edu

Abstract: Using the construction of D-branes with nonzero B field in the matrix model we give a physical interpretation of the known background independence in gauge theories on a noncommutative space. The background independent variables are identified as the degrees of freedom of the underlying matrix model. This clarifies and extends some recent results about the end point of tachyon condensation in D-branes with a B field. We also explain the freedom in the description which is parametrized by a two form Φ from the points of view of the noncommutative geometry on the worldvolume of the branes, and of the first quantized string theory.

Keywords: D-branes, M(atrix) Theories, Non-Commutative Geometry.

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Contents

1. Introduction

D-branes with nonzero B field have recently figured in different contexts in string theory. They were shown to arise in the matrix model [1] where an infinite collection of D0-branes can make higher Dp-branes [1, 2, 3]. This motivated the discovery of noncommutative geometry in string theory [4, 5], which was later studied by various authors including [6, 7]. More recently, following the work of Sen [8] the phenomenon of tachyon condensation on D-branes was studied by various people including [9, 16]. In particular, it became clear that by turning on a B field on D-branes, the analysis simplifies considerably [11, 12, 13, 16]. The main goal in this line of research is to learn about the ground state of the system after tachyon condensation. Since the open string states are expected to disappear, there should be no dependence on the background B field.

In this note we present a unified point of view on these problems which clarifies a number of issues. As in the matrix model, we will be thinking of the Dp-branes as built out of an infinite number of D0-branes. This perspective will make issues of background independence manifest. The matrix model variables are $N \times N$ matrices X^i. Different Dp-branes arise as different classical solutions $X^i_{cl} = x^i$ such that

$$[x^i, x^j] = i\theta^{ij} \,. \tag{1.1}$$

The degrees of freedom on the Dp-branes arise by expanding the dynamical variables around the classical solution

$$X^i = x^i + \theta^{ij}\widehat{A}_j \,, \tag{1.2}$$

and considering $\widehat{A}_j$ as functions of x^i. From this point of view it is clear that X^i are background independent and they do not depend on θ.

The subject of background independence has already appeared in the study of gauge theories on noncommutative space. Most of the discussion on this subject concerns with gauge theories on a torus, and is known in the mathematical literature as Morita equivalence. For the case of noncommutative $\mathbb{R}^p$ an explicit construction of background independent variables was given in [7, section 3.2]. We identify them with X^i of (1.2). The relation with the matrix model gives a physical interpretation of this mathematical observation.

Motivated by [16], this point of view also sheds light on the recently studied tachyon condensation phenomenon on Dp-branes. Here it is more convenient to use the IKKT matrix model [17], which is based on D1-branes. Their degrees of freedom X^i provide a background independent description of the Dp-branes. On the Dp-branes $[X^i, X^j] \neq 0$, but in the ground state without the Dp-branes $[X^i, X^j] = 0$. For the generic solution the $\mathrm{U}(N)$ symmetry ($N \to \infty$) of the matrix model is spontaneously broken to $\mathrm{U}(1)^N$. For the special solutions with $X^i = c^i$, which are proportional to the unit matrix the underlying $\mathrm{U}(N)$ symmetry is unbroken. As is standard in the IKKT model, we interpret the eigenvalues of X^i as spacetime points and identify the $\mathrm{U}(\infty)$ symmetry of the theory with the gauge symmetry on the Dp-branes.

We will also address a related topic associated with the freedom in the description of the theory parametrized by a choice of a two form Φ [18, 7]. For every background characterized by B, the closed string metric g, and the string coupling constant g_s we have a continuum of descriptions labeled by a choice of Φ. The open string metric G, the noncommutativity θ and the open string coupling G_s are determined by

$$\frac{1}{G+\Phi} + \theta = \frac{1}{g+B} G_s = g_s \left(\frac{\det(G+\Phi)}{\det(g+B)} \right)^{1/2} \tag{1.3}$$

(we have set $2\pi\alpha' = 1$). What is the geometric meaning of Φ? We argue that on a noncommutative space the choice of Φ enters through the commutator of two derivatives:

$$\begin{aligned} [x^i, x^j] &= i\theta^{ij} \\ [\partial_i, x^j] &= \delta_i^j \\ [\partial_i, \partial_j] &= -i\Phi_{ij} \, . \end{aligned} \tag{1.4}$$

Using these relations and the covariant derivative $D_i = \partial_i - i\widehat{A}_i$ we immediately get

$$[D_i, D_j] = -i(\widehat{F}_{ij} + \Phi_{ij}), \tag{1.5}$$

which suggests that if the action is written in terms of D_i, the dependence on $\widehat{F}$ and Φ should be only through the combination $\widehat{F} + \Phi$. Indeed, it was shown in [7] that the Dirac-Born-Infeld action depends only on this combination, and it was suggested that this could also be the case in higher orders.

From the string worldsheet point of view the freedom in Φ was interpreted in [7] as a choice of regularization which is related to field redefinition in spacetime. Following [19] we will present a calculation of the S-matrix which makes the assertion about the dependence on $\widehat{F}+\Phi$ manifest.

In the rest of this note we elaborate on these points. In section 2 we review the construction of Dp-branes in the matrix model and the analysis of the small fluctuations around the classical solution. In section 3 we discuss the relation between the underlying U(N) symmetry of the matrix model and the noncommutative gauge symmetry on the Dp-branes. In section 4 we use this physical picture to clarify some issues of background independence, and in section 5 we comment on some aspects of tachyon condensation. In section 6 we discuss various issues associated with Φ.

2. Dp-branes in matrix model

We consider the matrix model in flat $\mathbb{R}^{11}$ with transverse metric g_{IJ} ($I,J=1,\ldots,9$). We assume for simplicity that the metric g_{IJ} is block diagonal; i.e. it vanishes for $I=1,\ldots,p$, $J=p+1,\ldots,9$. The dynamical variables X^I are $N\times N$ hermitian matrices. The potential of the matrix model

$$-g_{IK}g_{JL}\,\mathrm{tr}[X^I,X^J][X^K,X^L] \tag{2.1}$$

determines the time independent equation of motion

$$g_{JL}[X^J,[X^K,X^L]]=0\,. \tag{2.2}$$

A simple set of a classical solutions are $X^I=x^I$ such that $[x^I,x^J]$ are proportional to the unit matrix in the $N\times N$ dimensional space (such solutions exist only for infinite N). Up to translations these can be taken to be $X^i=x^i$ for $i=1,\ldots,p$ (p is even) and $X^{a+p}=0$ for $a=1,\ldots,9-p$ with

$$[x^i,x^j]=i\theta^{ij}\,. \tag{2.3}$$

The rank of θ is p (if it is smaller, it means that effectively p is smaller) and we define the $p\times p$ matrix

$$B=\theta^{-1}\,. \tag{2.4}$$

We now expand the dynamical variables around the classical solution. Such an expansion was first performed to leading order in [3], and was later extended to all orders by many authors including [20]–[26]. Here we review the results. We change notation to

$$\begin{aligned} C_i &= B_{ij}X^j = B_{ij}x^j+\widehat{A}_i && i=1,\ldots,p\\ \phi^a &= \frac{1}{2\pi\alpha'}X^{a+p} && a=1,\ldots,9-p\,, \end{aligned} \tag{2.5}$$

and calculate

$$
\begin{aligned}
[C_i, C_j] &= -iB_{ij} + B_{ik}[x^k, \widehat{A}_i] - B_{jk}[x^k, \widehat{A}_j] + [\widehat{A}_i, \widehat{A}_j] \\
[C_i, \phi^a] &= B_{ij}[x^j, \phi^a] + [\widehat{A}_i, \phi^a] \,.
\end{aligned} \tag{2.6}
$$

Then the potential of the matrix model becomes

$$
\begin{aligned}
-g_{IK}g_{JL}\,\mathrm{tr}[X^I, X^J][X^K, X^L] &= (2\pi\alpha')^4 G^{ik}G^{jl}\,\mathrm{tr}\left(\widehat{F}_{ij} - B_{ij}\right)\left(\widehat{F}_{kl} - B_{kl}\right) - \\
&\quad -2(2\pi\alpha')^4 G^{ij} g_{ab}\,\mathrm{tr}\left(B_{ik}[x^k, \phi^a] + [\widehat{A}_i, \phi^a]\right)\left(B_{jl}[x^l, \phi^b] + [\widehat{A}_j, \phi^b]\right) - \\
&\quad -(2\pi\alpha')^4 g_{ac}g_{bd}\,\mathrm{tr}[\phi^a, \phi^b][\phi^c, \phi^d] \,,
\end{aligned} \tag{2.7}
$$

where tr is a trace over the N dimensional matrices and

$$
\begin{aligned}
G^{ij} &= -(2\pi\alpha')^{-2}\theta^{ik}g_{kl}\theta^{lj} \\
\widehat{F}_{ij} &= -iB_{ik}[x^k, \widehat{A}_j] + iB_{jk}[x^k, \widehat{A}_i] - i[\widehat{A}_i, \widehat{A}_j] \,.
\end{aligned} \tag{2.8}
$$

If x^i generate the entire set of $N \times N$ matrices, every $N \times N$ matrix M can be expressed as a function of x^i. We thus generate the space of the brane as parametrized by x^i. Since the x^i do not commute, the brane is noncommutative. A convenient ordering of x^i in $M(x^i)$ is Weyl-ordering.[1] If M_1 and M_2 are ordered properly, the ordered product is the star product $M_1 * M_2$ with the noncommutativity parameters θ. If x^i do not generate the whole set of $N \times N$ matrices, every $N \times N$ matrix M can be expressed as a $K \times K$ matrix whose entries are functions of x^i. For simplicity we will consider the case of finite K.

It is easy to see that for every $K \times K$ matrix of functions $M(x^i)$

$$
\begin{aligned}
\partial_i M &= -iB_{ik}[x^k, M] \\
D_i M &= \partial_i M - i[\widehat{A}, M] = -i[C_i, M] \,,
\end{aligned} \tag{2.9}
$$

where x^i is proportional to the unit matrix in the K dimensional space. In (2.9) all the products are matrix products and $*$ products of their elements.

Using this way of representing the matrices, $\widehat{A}_i$ and ϕ^a can be regarded as $K \times K$ matrices which are functions of x^i. The potential V (2.1) can now be written as

$$
\begin{aligned}
-g_{IK}g_{JL}\,\mathrm{tr}[X^I, X^J][X^K, X^L] = (2\pi\alpha')^4 \int \frac{d^p x}{(2\pi)^{\frac{p}{2}}} \mathrm{Pf}B G^{ik}G^{jl}\mathrm{tr}\left(\widehat{F}_{ij} - B_{ij}\right)\left(\widehat{F}_{kl} - B_{kl}\right) + \\
+ 2G^{ij}g_{ab}\mathrm{tr}D_i\phi^a D_j\phi^b - g_{ac}g_{bd}\mathrm{tr}[\phi^a, \phi^b][\phi^c, \phi^d] \,.
\end{aligned} \tag{2.10}
$$

[1] Weyl-ordering can be defined in terms of a power series in x^i, with each monomial averaged over the ordering of the factors. Alternatively, it is defined in terms of a Fourier transform with $e^{ip_i x^i}$ ordered as a power series in $p_i x^i$.

Here tr is a trace over the $N \times N$ matrices and tr is a trace over the $K \times K$ matrices. The measure of integration is determined as in the relation between a trace over the Hilbert space and the integral over the classical phase space

$$\mathrm{tr} = \int \frac{d^p x}{(2\pi)^{\frac{p}{2}}} \frac{1}{\mathrm{Pf}\theta} \mathrm{tr} = \int \frac{d^p x}{(2\pi)^{\frac{p}{2}}} \mathrm{Pf} B \mathrm{tr} \,. \tag{2.11}$$

It is straightforward to add the kinetic term of X^I and the fermions and to write them as an integral over the brane. The whole action becomes then the minimal coupling in noncommutative rank K super-Yang-Mills theory.

We would like to make a number of comments:

1. Using (2.11) on the unit operator we deduce that $N/V = \mathrm{Pf} BK/(2\pi)^{p/2}$, where V is the volume of the brane. Therefore, we can interpret B as the background B field on the Dp-brane and N as the induced D0-brane charge on the Dp-brane. We learn that these Dp-branes necessarily have B field on them.

2. Restoring factors of α' in (1.3) we get

$$\begin{aligned} \frac{1}{G + 2\pi\alpha'\Phi} + \frac{\theta}{2\pi\alpha'} &= \frac{1}{g + 2\pi\alpha' B} \\ G_s &= g_s \left(\frac{\det(G + 2\pi\alpha'\Phi)}{\det(g + 2\pi\alpha' B)} \right)^{1/2} . \end{aligned} \tag{2.12}$$

   A natural choice of Φ, which was identified in [7] is

$$\begin{aligned} \Phi &= -B \\ \theta &= \frac{1}{B} \\ G &= -(2\pi\alpha')^2 B \frac{1}{g} B \\ G_s &= g_s \det(2\pi\alpha' B g^{-1})^{1/2} \,. \end{aligned} \tag{2.13}$$

   These expressions look like the expressions in the zero slope limit, but they are exact for all values of α'. The form of the metric *proportionalto*naturally leads to this choice of Φ. In section 6 we will explain why the matrix model leads to this choice.

3. The matrix model is based on a certain scaling limit of string theory [27, 28], which is essentially a zero slope limit, in which the lagrangian simplifies and is quadratic in the field strength. However, for the manipulations in this section this limit is not essential. We could have repeated the analysis replacing the lagrangian with the full lagrangian of X in string theory, ending with a more complicated answer in terms of the same dynamical fields on the Dp-brane.

4. We can repeat the calculation for the IKKT model which is based on D1-branes [17]. Then the entire Lagrangian is given by (2.1) (plus fermion terms), and there is no need to include a kinetic term. Unlike the BFSS model, here we are not aware of a zero slope limit which justifies the use of the minimal lagrangian. However, as in the case of D0-branes, the inclusion of higher order terms do not affect our main conclusion.

3. The gauge symmetry

The U(N) gauge symmetry of the matrix model acts on the dynamical variables as $\delta X^I = i[\lambda, X^I]$. If we do not want it to act on the background x^i, it must act as a noncommutative gauge transformation of rank K on the D-brane [3]–[25]:

$$\begin{aligned} \delta \widehat{A}_i &= iB_{ik}[\lambda, x^k] + i[\lambda, \widehat{A}_i] = \partial_i \lambda + i[\lambda, \widehat{A}_i] \\ \delta \phi^a &= i[\lambda, \phi^a] \,, \end{aligned} \tag{3.1}$$

where again the products are matrix products and $*$ products of the elements. We conclude that the noncommutative gauge symmetry on the Dp-brane is the underlying U(N) symmetry of the matrix model.

For sufficiently small $\theta^{ij}\widehat{A}_j$ all the eigenvalues of the matrix $\partial_i X^j(x)$ are positive and we can take X^i as coordinates on the brane [22, 29]. The advantage of doing that is that X^i are background independent and can be used to describe the coordinates on branes for all values of θ. This explains the origin of the ordinary gauge fields on the branes [22, 29] by noting that

$$\left[-\frac{1}{2}B_{ij}X^j + A_i(X)\right]\partial_l X^i = -\frac{1}{2}B_{lj}x^j + \frac{1}{2}\partial_l(x^j \widehat{A}_j(x)) \,. \tag{3.2}$$

Here $A_i(X)$ are ordinary commutative gauge fields with ordinary gauge symmetry which are related to the noncommutative gauge fields through [7]

$$\widehat{A}_i = A_i + \frac{1}{2}\theta^{ab}(2A_b\partial_a A_i + A_a \partial_i A_b) + \mathcal{O}(\theta^2) \tag{3.3}$$

and the x's are treated as commuting coordinates. (The order θ^2 corrections to (3.2) were verified by Govindan Rajesh [30]). From (3.2) we readily get

$$[B_{ij} + F_{ij}(X)]\,\partial_k X^i \partial_l X^j = B_{kl} \,. \tag{3.4}$$

This can be interpreted as the statement that the coordinates x^i are defined in terms of X^i as the coordinates in which the commutative field strength on the brane is constant.

This discussion makes it clear that while the noncommutative description is always valid, the commutative description with the coordinates X^i cannot be used when $\partial_i X^j(x)$ is not positive definite. This is the case when the fluctuations in X around the classical solution x are large or when $\widehat{F}$ is sufficiently large. This shows that the change of variables from $\widehat{A}$ and A has only a finite radius of convergence. More precisely, we see from (3.4) that when $\partial_i X^j(x)$ has a zero eigenvalue, F must diverge. Similarly, when $B+F$ has a zero eigenvalue $\partial_i X^j(x)$ must diverge. A first sign of such a pole has already appeared in [7], where it was found that the expression for F as a function of $\widehat{F}$ has a pole for constant $\widehat{F}$ when $\widehat{F}-1/\theta$ has a zero eigenvalue, and that $\widehat{F}$ diverges when $F+B$ has a zero eigenvalue. Here we see the same fact for any F which is not necessarily constant.

4. Background independence

Section 3.2 of [7] presents an explicit version of Morita equivalence for the special case of noncommutative $\mathbb{R}^p$ and discusses it as background independence. In background independence we mean the following. We hold the closed string metric g and the closed string coupling g_s fixed and vary the background B on the D-branes. This has the effect of changing the noncommutativity θ, as well as the open string metric G and the open string coupling G_s.

This background independence should not be confused with the choice of Φ in (1.3). The later keeps the background fixed and changes the language used to describe it.

Let us first review and extend the discussion in [7] for the special case of θ of rank p. We use the coordinates x^i satisfying $[x^i, x^j] = i\theta^{ij}$ to form the objects

$$\begin{aligned} C_i &= \theta^{-1}_{ij} x^i + \widehat{A}_i \\ X^i &= \theta^{ij} C_j = x^i + \theta^{ij}\widehat{A}_j \,, \end{aligned} \tag{4.1}$$

which satisfy

$$\begin{aligned} -i[C_i, C_j] &= \widehat{F}_{ij} - \theta^{-1}_{ij} \\ i[X^i, X^j] &= \theta^{ik}\widehat{F}_{kl}\theta^{lj} - \theta^{ij} \,. \end{aligned} \tag{4.2}$$

The main point is that the expression for $\widehat{F}$ does not involve explicit derivatives with respect to x.

We define

$$\partial'_i = \partial_i + i\theta^{-1}_{ij} x^j \,, \tag{4.3}$$

and write the covariant derivative as

$$D_i = \partial'_i - iC_i(x) \,. \tag{4.4}$$

Using

$$[\partial'_i, x^j] = 0 \tag{4.5}$$

we find that for every function of x, $M(x)$

$$[D_i, M(x)] = -i[C_i(x), M(x)]\,. \tag{4.6}$$

Therefore, using (4.2) and (4.6), the entire lagrangian can be expressed in terms of $C(x)$ and perhaps other fields $M(x)$, without explicit derivatives with respect to x.

The minimal $\widehat{F}^2$ lagrangian can be replaced by $(\widehat{F} - \theta^{-1})^2$, which differs from it by a constant and a total derivative. More generally, in string theory the Dirac-Born-Infeld action was shown in [7] to depend only on the combination $\widehat{F} + \Phi$, which becomes with the natural choice (2.13) $\widehat{F} - \theta^{-1}$.

The most general lagrangian in which $\widehat{F}$ and θ appear only in the combination $\widehat{F} - \theta^{-1}$ can be written in terms of C and perhaps other fields $M(x)$ without explicit θ dependence

$$\frac{1}{G_s}\int d^px\sqrt{G}\mathrm{tr}\mathcal{L}(G, C(x), M(x))\,. \tag{4.7}$$

θ enters only in the commutation relations of x^i.

Now we must make sure that as we vary the background we hold g and g_s fixed rather than G and G_s. In other words, when the Lagrangian is expressed in terms of g, g_s and θ, all the θ dependence is in the choice of the commutation relations of x^i. This is not the case when the lagrangian is expressed, as in (4.7), in terms of the metric G. But because of the simple form of (2.13) this can easily be fixed by expressing the lagrangian in terms of the variables X^i and the metric g_{ij}. For example, the quartic terms can be written as $G^{ii'}G^{jj'}[C_i, C_j][C_{i'}, C_{j'}] \sim g_{ii'}g_{jj'}[X^i, X^j][X^{i'}, X^{j'}]$. We end up with a lagrangian of the form

$$\frac{1}{G_s}\int d^px\sqrt{G}\mathrm{tr}\mathcal{L}(g_{ij}, X^i(x), M(x)) \sim \frac{1}{g_s}\int d^px\frac{1}{\mathrm{Pf}\theta}\mathrm{tr}\mathcal{L}(g_{ij}, X^i(x), M(x)). \tag{4.8}$$

Note that the measure in the last expression does not have a factor of $\sqrt{g}$; instead it has a factor of $\frac{1}{\mathrm{Pf}\theta}$ which transforms the same way under linear transformations of the x's. We also note that the metric g is used to contract the indices of the fields X, while the metric G and the noncommutativity θ transform with the coordinates x.

From (4.8) it is clear that changes in θ can be compensated by changes in x without changes in X, the metric g and the closed string coupling g_s. This proves background independence.

The construction of the D-branes in the matrix model gives a simple physical interpretation of this result. The matrix model does not enjoy full background independence. As a formulation of the theory in the lightcone frame, it is invariant only under a few changes of the background. But since D-branes are solutions of the stationary equation of motion of the model (2.2), the latter is invariant under changes of the background, which affect only their details. An example of such a change is a change of the B field on the D-brane, holding the metric g fixed. Therefore, it

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is not surprising that the matrix model leads to a formulation of D-branes which is invariant under changes of θ for fixed g.

It is now obvious why the theory should be expressed in terms of X^i. These are the original dynamical variables of the matrix model which are manifestly background independent. Different values of θ correspond to different classical solutions for the same degrees of freedom with the same Lagrangian. The matrix model also makes it clear that the indices of the coordinates on the branes x^i are the same as in the open string metric G_{ij} and in the noncommutativity θ^{ij}, and that they are distinct from the indices of the spacetime coordinates X^i which are the same as those in the metric g_{ij}. Finally, we note that the measure in (4.8) is the same as in (2.11), or in any generalization of it including higher order terms which are higher than quartic in X.

5. Tachyon condensation

The work of Sen [8] and his followers on tachyon condensation in open string theory (space filling D-branes) was simplified and extended by turning on a nonzero B field [11, 12, 13, 16]. It was convincingly argued that the tachyon rolls to the closed string vacuum and the solitons in this vacuum were identified with D-branes.

Our discussion based on the matrix model cannot be directly applied in this case for several reasons. First, this problem is not supersymmetric, and correspondingly, the theory on the space filling D-branes has tachyons T, which are not present in the matrix model. Second, to fully use the simplification due to the B field one should turn on B of maximal rank including along the time direction (now p includes the time direction). In this case we cannot use the zero slope limit which simplifies the effective lagrangian and makes it quadratic in the field strength.

However, several lessons can still be drawn as above (A connection between the matrix model and this problem was anticipated in [12, 31]). The effective lagrangian on a single D-brane is of the general form (we set $2\pi\alpha' = 1$ and use euclidean signature)

$$\frac{1}{G_s}\int d^px\left[V(T)\sqrt{\det(G_{ij}+\widehat{F}_{ij}+\Phi_{ij})}+\sqrt{G}f(T)G^{ij}D_iTD_jT+\cdots\right]. \tag{5.1}$$

We use the convenient choice $\Phi = -B$ (2.13), and write it in terms of our variables $C_i = B_{ij}x^j + \widehat{A}_i$ and $X^i = \theta^{ij}C_j$ [32]

$$\begin{aligned}&\frac{1}{G_s}\int d^px\left[V(T)\sqrt{\det(G_{ij}+i[C_i,C_j])}-\sqrt{G}f(T)G^{ij}[C_i,T][C_j,T]+\cdots\right]=\\&\frac{\det B}{G_s}\int d^px\left[V(T)\sqrt{\det(g^{ij}+i[X^i,X^j])}-\frac{1}{\sqrt{g}}f(T)g_{ij}[X^i,T][X^j,T]+\cdots\right]=\\&\frac{(2\pi)^{\frac{p}{2}}}{g_s}\,\mathrm{tr}\left[V(T)\sqrt{\det(\delta_i^j+ig_{ik}[X^k,X^j])}-f(T)g_{ij}[X^i,T][X^j,T]+\cdots\right].\end{aligned} \tag{5.2}$$

In the last expression we represented X^i and T as $N \times N$ matrices in the $N \to \infty$ limit. We see that, as expected, when the action is expressed in terms of the closed strings metric g, the string coupling g_s, and the dynamical variables X^i, rather than in terms of G and G_s, there is no dependence on θ or B (this is not the case when the action is expressed in terms of C_i). This is a general result which persists even in the higher order terms, as it follows simply from the general properties of $X^i = x^i + \theta^{ij}\widehat{A}_i$ in noncommutative gauge theories, or from our underlying matrix model interpretation.

Assuming that $V(T)$ has a unique minimum T_c, at the vacuum $T = T_c$ is proportional to the unit matrix. The vacuum is commutative and is characterized by $X^i = X^i_c$ satisfying

$$[X^i_c, X^j_c] = 0. \tag{5.3}$$

Depending on the eigenvalues of X^i_c the full $\mathrm{U}(N)$ symmetry is broken to $\mathrm{U}(N_1) \times \mathrm{U}(N_2)\cdots$. In one extreme case it is $\mathrm{U}(1)^N$, and in the other, when all X^i_c are proportional to the unit matrix, $\mathrm{U}(N)$ is unbroken.

In a recent paper [16] it was suggested that the classical solution in the ground state has $X^i_c = 0$. We add to this the other solutions satisfying (5.3). These include other $\mathrm{U}(N)$ invariant solutions, where X^i_c are all proportional to the unit matrix and other solutions which break the $\mathrm{U}(N)$ symmetry. Motivated by our discussion in the previous sections, we also add the interpretation of the $\mathrm{U}(N)$ symmetry as the symmetry of an underlying matrix model. Correspondingly, we interpret the eigenvalues of the classical solutions X^i_c as N spacetime points. Section 3 makes it clear that the $\mathrm{U}(N)$ symmetry is not bigger than the rank K noncommutative gauge symmetry on the D-branes. In fact, these two symmetries are the same symmetry. Finally, the discussion in section 4 of background independence makes the fact that the answers are independent of θ manifest.

6. Comments on Φ

As we mentioned in the introduction, we interpret Φ as appearing in the defining commutation relations (1.4)

$$\begin{aligned} [x^i, x^j] &= i\theta^{ij} \\ [\partial_i, x^j] &= \delta^j_i \\ [\partial_i, \partial_j] &= -i\Phi_{ij}\,. \end{aligned} \tag{6.1}$$

We assume, for simplicity, that θ is of maximal rank and define, as in (4.3) [7]

$$\partial'_i = \partial_i + i\theta^{-1}_{ij}x^j\,, \tag{6.2}$$

which satisfy

$$\begin{aligned}
[\partial'_i, x^j] &= 0 \\
[\partial'_i, \partial'_j] &= -i(\Phi_{ij} + \theta^{-1}_{ij}).
\end{aligned} \tag{6.3}$$

The covariant derivatives

$$D_i = \partial_i - i\widehat{A}_i = \partial'_i - iC_i(x) \tag{6.4}$$

satisfy

$$[D_i, D_j] = -i(\widehat{F}_{ij} + \Phi_{ij}). \tag{6.5}$$

Three special cases are of particular importance.

1. $\theta = 0$. This is the commutative theory. Here we cannot define ∂', but on the other hand, we can define $\widetilde{\partial}_i = \partial_i - \frac{i}{2}\Phi_{ij}x^j$ such that

$$\begin{aligned}
[\widetilde{\partial}_i, x^j] &= \delta_i^j \\
[\widetilde{\partial}_i, \widetilde{\partial}_j] &= 0.
\end{aligned} \tag{6.6}$$

   Physically, this is the familiar case of electrons in background magnetic field Φ, and $\widetilde{\partial}$ is the ordinary derivative.

2. $\Phi = 0$. This is the ordinary noncommutative theory whose derivatives commute.

3. $\Phi = -\theta^{-1}$. In this case the derivatives ∂' commute both with the coordinates x and with themselves. The phase space generated by x^i and ∂_i is degenerate and we can set ∂'_i to zero. In other words, the algebra is the same as with $x^i = i\theta^{ij}\partial_j$. If we consider a quantum system based on this phase space, the wave functions are not functions of all the x's but only of half of them. Physically, this choice arises in the problem of electrons constrained to be in the first Landau level, where the wave functions depend only on half of the coordinates.

Now it is clear why in the matrix model we automatically found the third choice $\Phi = -\theta^{-1}$. In that problem we started with D0-branes and created the worldvolume of the Dp-branes. Derivatives on that worldvolume appeared, as in (2.9), as commutators with the coordinates. Therefore, the coordinates and the derivatives are not independent and a linear combination of them ∂' must be set to zero.

We would like to end this section about Φ by commenting about the way it arises in the first quantized description of string theory. It was suggested in [7] that the freedom in Φ is associated with a choice of regularization in the worldsheet path integral. This point was made more explicit in [19]. Setting $2\pi\alpha' = 1$ the propagator of the worldsheet fields X along the boundary is given by [33]–[34]

$$\langle X^i(\tau)X^j(0)\rangle = -\frac{1}{2\pi}G_0^{ij}\log\tau^2 + \frac{i}{2}\theta_0^{ij}\epsilon(\tau)\,, \tag{6.7}$$

where

$$\frac{1}{G_0} = \left(\frac{1}{g+B}\right)_S$$
$$\theta_0 = \left(\frac{1}{g+B}\right)_A . \tag{6.8}$$

It was important in the discussion in [7] that the second term in the propagator affects correlation functions of vertex operators at separated points $\langle\cdots\rangle_{g,B}$ only in a prefactor

$$\langle\cdots\rangle_{g,B} = e^{-\frac{i}{2}\sum_{n>m} k_i^n \theta_0^{ij} k_j^m \epsilon(\tau_n-\tau_m)} \langle\cdots\rangle_{G_0,0} , \tag{6.9}$$

where k^n are the spacetime momenta of the operators. This observation led to the conclusion that if the action is written in terms of G_0, θ_0 and G_s, the noncommutativity θ_0 enters only in the star product.

Let us examine what happens as we vary Φ, G and θ while holding g and B fixed. The symmetric and antisymmetric parts of

$$\frac{1}{G+\Phi} + \theta = \frac{1}{g+B} \tag{6.10}$$

are

$$\left(\frac{1}{G+\Phi}\right)_S = \frac{1}{G_0}$$
$$\left(\frac{1}{G+\Phi}\right)_A = -\theta + \theta_0 , \tag{6.11}$$

and therefore

$$\langle\cdots\rangle_{G,\Phi} = e^{-\frac{i}{2}\sum_{n>m} k_i^n (\theta_0-\theta)_0^{ij} k_j^m \epsilon(\tau_n-\tau_m)} \langle\cdots\rangle_{G_0,0} \tag{6.12}$$

and

$$\langle\cdots\rangle_{g,B} = e^{-\frac{i}{2}\sum_{n>m} k_i^n \theta^{ij} k_j^m \epsilon(\tau_n-\tau_m)} \langle\cdots\rangle_{G,\Phi} . \tag{6.13}$$

We now add Chan-Paton factors to the external legs. Each order of these factors is associated with an order of the τ_n's. To find the S-matrix for a specific order of the Chan-Paton factors we integrate over the τ_n's in a given order. We find a product of the standard phase of noncommutative theories $\exp(-\frac{i}{2}\sum_{n>m} k_i^n \theta^{ij} k_j^m)$ with the parameter θ, and another factor which depends only on G and Φ. The latter factor is the same as in a commutative theory ($\theta = 0$) with closed string metric G and B field given by Φ. We stress that the S-matrix is universal and does not depend on a choice of regularization. This fact is usually proven by going to a convenient region in momentum space where the contribution from the boundary of the integration region vanishes.

We now look for an effective action which reproduces this S-matrix. Here we face the known ambiguity associated with field redefinition. Ignoring the phase factor

$\exp(-\frac{i}{2}\sum_{n>m} k_i^n \theta^{ij} k_j^m)$, we can use the ordinary action which depends only on the metric G and the combination $F+\Phi$. (Here is where we make a choice because there are also other actions which lead to the same S-matrix.) The phase is then added by making the theory noncommutative and turning F into $\widehat{F}$. We end up with an action which depends on Φ only through the combination $\widehat{F}+\Phi$ and on θ only through the star product. Finally, the expression for the open string coupling (1.3) is determined by evaluating the action for $\widehat{F}=0$. This completes the proof of the conjectured form of the action [7].

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